A PROBLEM OF TRANSIENT HEAT CONDUCTION IN A SEMIBOUNDED BODY WITH AN INTERNAL CYLINDRICAL HEAT SOURCE

I. A. Ioffe

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A solution is obtained to the problem of transient heat conduction in a semibounded body both with an internal cylindrical heat source and with constant temperatures at both body and cylinder surfaces. Expressions are derived for the temperature field and the rate of heat flow.

It is required to solve the differential equation of heat conduction within the region $y \geq 0$ (Fig. 1) bounded inside by an infinitely long circular cylindrical surface with radius ρ and the axis parallel to plane $y = 0$, with the following constraints:

$$t(x,\ y,\ 0) = t_0, \tag{1}$$

$$t(x,\ 0,\ \tau) = t_0, \tag{2}$$

$$t(x,\ \infty,\ \tau) = t_0, \tag{3}$$

$$t(x,\ y,\ \tau)\big|_{x,\ y \in \Gamma} = t_\text{т}, \tag{4}$$

where Γ is the circular contour with radius ρ, and with constant thermophysical properties of the medium within the region $y \geq 0$.

An engineering analog of this problem is the transient heat transfer from an underground pipeline with a liquid or gaseous heat carrier to the surrounding soil. The steady-state temperature field and the rate of heat flow are in this case described by the Forchheimer equations [1]. The solution to the simplified version of this problem (isothermal cylinder in an infinitely large body) is well known [2, 3]. As to integrating the equation of heat conduction with the constraints (1)-(4), this has been done in [4]. In order to arrive at the results in [4], it is necessary to have tabulated first the values of an intricate function of two variables.

Temperature Field. The well-known relation in [3, 5] describes the temperature field in an infinitely large body due to a momentary cylindrical surface heat source. Accordingly, the integral with respect to time in this expression will represent the temperature distribution due to a constant cylindrical heat source. We will now examine the following function which represents, in dimensionless variables, a superposition of such a source and a mirror-symmetrical sink on the other side of the x-axis, the power of both varying with time:

$$\theta(r,\ R,\ \text{Fo}) = \frac{1}{4\pi}\int_0^{\text{Fo}} \varphi(u)\left[\exp\left(-\frac{r^2+1}{4(\text{Fo}-u)}\right) I_0\left(\frac{r}{2(\text{Fo}-u)}\right) - \exp\left(-\frac{R^2+1}{4(\text{Fo}-u)}\right) I_0\left(\frac{R}{2(\text{Fo}-u)}\right)\right]\frac{du}{\text{Fo}-u}, \tag{5}$$

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where

$$r = \frac{1}{\rho}\sqrt{x^2 + (y_0' - y)^2}, \qquad R = \frac{1}{\rho}\sqrt{x^2 + (y_0' + y)^2}.$$

and the dimensionless radius-vectors r, R in Cartesian coordinates are

$$\theta(r,\ R,\ \mathrm{Fo}) = \frac{t(x,\ y,\ \tau) - t_0}{t_\mathrm{T} - t_0}, \qquad \mathrm{Fo} = \frac{a\tau}{\rho^2}$$

We note that expression (5), which represents the linear sum of integrals in the equation of heat conduction, satisfies this equation and, with a change to dimensionless variables, also the conditions (1)-(3). It remains now to determine the unknown function $\varphi(\mathrm{Fo})$ so as to satisfy the boundary condition (4). Then, according to the unicity theorem, expression (5) will be the solution to the given constraint problem.

Condition (4) can also be written as

$$\theta(1,\ R_1,\ \mathrm{Fo}) = 1, \tag{6}$$

where $R_1 = R|_{r=1}$. Since $2y_0 - 1 \le R_1 \le 2y_0 + 1$, hence condition (6) can be realized only approximately – more accurately as the ordinate y_0 increases relative to the radius of the heat source.* We will assume that y_0 is sufficiently large to make

$$\theta(1,\ R_1,\ \mathrm{Fo}) \approx \theta(1,\ 2y_0,\ \mathrm{Fo}) = 1. \tag{7}$$

Referring (5) to points on the circle with radius r = 1 and considering conditions (7), we obtain the following Volterra integral equation of the first kind of convolution:

$$1 = \frac{1}{4\pi}\int_0^{\mathrm{Fo}} \varphi(u)\left[\exp\left(-\frac{1}{2(\mathrm{Fo}-u)}\right) I_0\left(\frac{1}{2(\mathrm{Fo}-u)}\right) - \exp\left(-\frac{4y_0^2+1}{4(\mathrm{Fo}-u)}\right) I_0\left(\frac{y_0}{\mathrm{Fo}-u}\right)\right]\frac{du}{\mathrm{Fo}-u}. \tag{8}$$

Applying the Laplace transformation to both sides of Eqs. (5) and (8), and applying the convolution theorem, we have

$$\bar{\theta}(r,\ R,\ s) = \frac{\bar{\varphi}(s)}{4\pi}\left[2I_0(\sqrt{s})K_0(r\sqrt{s}) - 2I_0(\sqrt{s})K_0(R\sqrt{s})\right], \tag{9}$$

$$\frac{1}{s} = \frac{\bar{\varphi}(s)}{4\pi}\left[2I_0(\sqrt{s})K_0(\sqrt{s}) - 2I_0(\sqrt{s})K_0(2y_0\sqrt{s})\right], \tag{10}$$

where $\bar{\theta}$(r, R, s) and $\bar{\varphi}$(s) are the transforms of the respective functions.

A simultaneous solution of Eqs. (9) and (10) yields

$$\bar{\theta}(r,\ R,\ s) = \frac{K_0(r\sqrt{s}) - K_0(R\sqrt{s})}{s\left[K_0(\sqrt{s}) - K_0(2y_0\sqrt{s})\right]}. \tag{11}$$

The original function θ(r, R, Fo) is found by the Melin inversion formula. In order to converge the complex integral to a real one, we use a well-known contour of the Hankel type which joins the branch points s = 0 and s = ∞ of the integrand function with a cut along the negative imaginary semiaxis (see, e.g., Figs. 15 and 12, [5]). Continuing further by the standard procedure of integration in a complex plane and by using the properties of Bessel functions, we find

$$\theta(r,\ R,\ \mathrm{Fo}) = \frac{\ln(R/r)}{\ln 2y_0} - \frac{2}{\pi}\int_0^\infty e^{-z^2\mathrm{Fo}} \times \frac{[J_0(z) - J_0(2y_0z)][Y_0(rz) - Y_0(Rz)] - [J_0(rz) - J_0(Rz)][Y_0(z) - Y_0(2y_0z)]}{[J_0(z) - J_0(2y_0z)]^2 + [Y_0(z) - Y_0(2y_0z)]^2}\frac{dz}{z}. \tag{12}$$

Rate of Heat Flow. In the sense of assumption (7), the sink field is constant within the surface region r = 1. The total field in this region becomes then cylindrical and the rate of heat flow from a unit of source

* As a rule, pipelines are buried at depths much greater than their radius.

length is

$$Q = \left(-2\pi r\lambda \frac{\partial t}{\partial r}\right)_{r=1} = -2\pi\lambda (t_T - t_0) \frac{\partial \theta}{\partial r}\bigg|_{r=1}. \tag{13}$$

We also note that, since $R|_{r=1} = 2y_0$ has been assumed in (7),

$$\frac{\partial R}{\partial r}\bigg|_{r=1} = 0. \tag{14}$$

Differentiating (12), taking into account (14) and bearing in mind that [6]

$$z\,[J_1(nz)\,Y_0(nz) - J_0(nz)\,Y_1(nz)] = \frac{2}{\pi n},$$

we obtain

$$q = \frac{2\pi}{\ln 2y_0} + 4\int\limits_0^\infty e^{-z^2\,\mathrm{Fo}}\,\frac{2/(\pi z) + J_0(2y_0z)\,Y_1(z) - J_1(z)\,Y_0(2y_0z)}{[J_0(z) - J_0(2y_0z)]^2 + [Y_0(z) - Y_0(2y_0z)]^2}\,dz, \tag{15}$$

where $q = Q/[\lambda(t_T - t_0)]$ is the dimensionless rate of heat flow.

Estimating the Error of the Solution. An elementary analysis will show that the integrals in (12) and (15) will tend to zero as $\mathrm{Fo} \to \infty$. Consequently, the expressions

$$\theta_{ss} = \frac{\ln(R/r)}{\ln 2y_0}, \quad q_{ss} = \frac{2\pi}{\ln 2y_0} \tag{16}$$

represent the steady-state components of the dimensionless temperature and heat-flow rate, respectively. As has been indicated earlier, this problem has an exact steady-state solution (Forchheimer equations). These formulas can be expressed in our notation as follows:

$$\theta^*_{ss} = \frac{\ln(R/r)}{\ln\left(y_0 + \sqrt{y_0^2 + 1}\right)}, \quad q^*_{ss} = \frac{2\pi}{\ln\left(y_0 + \sqrt{y_0^2 + 1}\right)}. \tag{17}$$

The discrepancy between (16) and (17), quantitatively negligible when $y_0^2 \gg 1$, is explained by the inexactness of condition (7). The relative error δ in the dimensionless temperature and in the dimensionless heat-flow rate according to calculations, is very small already at small values of y_0 and decreases fast with increasing y_0. Thus, we have $\delta = 0.015$ at $y_0 = 3$ and $\delta = 0.004$ at $y_0 = 5$.

A correct evaluation of the error in the transient solution is very difficult. For a rough estimate, however, we will utilize the following features in the trend of functions (12) and (15).

1. From the relation between the original and its transform

$$\theta(r, R, 0) = \lim_{s \to \infty} s\bar{\theta}(r, R, s)$$

and with the asymptotic formula for MacDonald functions

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\, e^{-z},$$

one can find that

$$\theta(r, R, 0) = \begin{cases} 1 & r = 1, \\ 0 & r > 1. \end{cases}$$

Thus, Eq. (12) and, therefore, also its partial derivative (15) constitute the exact solution to the problem at the time $t = 0$, because the constraints are satisfied precisely when $\mathrm{Fo} = 0$. The physical meaning of this result is that the effect of a heat source and a heat sink on the temperature at their contours, and vice versa, is negligible after short periods of time. The dimensionless temperature at the source contour departs from unity, on the other hand, after a period of time when the temperature function of the sink becomes appreciably different on the source contour at different distances from the sink center.

2. Solutions (12) and (15) as well as certain yet unknown functions representing the exact solution to the problem are, from the physical viewpoint, monotonic (in the narrow sense) functions on the interval $0 < \mathrm{Fo} < \infty$.

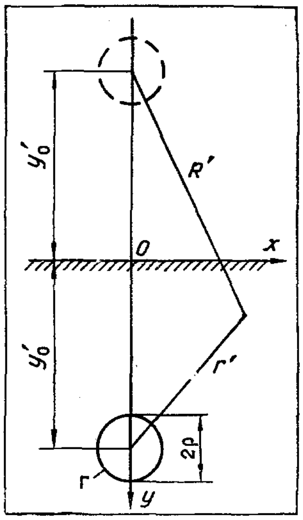

Fig. 1. Scheme of problem.

Thus, (12) and (15), which are the exact solution to the problem for Fo = 0, contain some relative error when Fo = ∞. On the basis of the additional consideration (in the paragraph on the rate of heat flow), it may be assumed that the relative error is maximum in the steady-state case and does not exceed the error on the interval $0 < Fo < \infty$.

NOTATION

t	is the temperature;
t_0	is the initial temperature;
t_T	is the temperature at the surface of a cylindrical heat source;
x, y	are the linear space coordinates;
y_0'	is the ordinate of the source center;
ρ	is the radius of the source;
r, R	are the dimensionless radius-vectors of a field point;
y_0	is the ordinate of the source center, with ρ as the characteristic dimension;
τ	is the time;
φ(Fo)	is the power of heat source and heat sink;
λ	is the thermal conductivity;
a	is the thermal diffusivity;
Q	is the rate of heat flow from unit of source length;
q	is the dimensionless rate of heat flow from unit of source length;
θ_{ss}, q_{ss}	are the steady-state components of the dimensionless temperature and the dimensionless heat-flow rate, respectively

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